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Mathematical basis for analysis of partially coherent wave propagation in nonlinear, non-instantaneous, Kerr media

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Abstract

A general mathematical basis for analyzing propagation of partially coherent light in nonlinear, non-instantaneous, Kerr media is presented. The presentation unifies existing approaches, but also provides the generalizations necessary for an analysis of more complicated wave fields than those previously considered. In particular, it is demonstrated how to generalize the analysis to situations where the light field originates from several sources, which do not necessarily have the same stochastic properties.

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1. Introduction

Nonlinear wave propagation is an important topic in a wide range of very diverse physical areas, e.g. in optics and plasma physics as well as in Bose–Einstein condensates, and in particular the possibility of soliton formation has attracted strong interest over many years [1]. Until quite recently, the commonly held impression was that solitons are inherently coherent structures with well-defined amplitudes and phases, existing as a stable balance between two counteracting effects, e.g. dispersion and self focusing. However, in the middle of the 1990s, it was demonstrated that partially coherent and even incoherent spatial optical solitons are also possible [2–7]. More specifically, partially coherent bright optical solitons were found to exist in nonlinear, non-instantaneous media such as biased photo-refractive crystals where the response of the medium is so slow that it only reacts on the time-averaged intensity.

With the first experimental observation of partially coherent solitons [3], a new branch of research within the field of nonlinear optics was initiated and has subsequently attracted strong interest. Theoretical analysis of the dynamics of partially coherent light involves a stochastic description of the field envelope. Specifically, as was shown in a pioneer paper by Pasmanik [2], a self-consistent description of the problem is possible using the auto-correlation function (CF) of the field envelope. More recently the Wigner function (WF) approach [8] was also suggested for studies of the same problem. The Wigner function is unambiguously related to the correlation function by being simply its Fourier transform and therefore the CF and Wigner approaches are completely equivalent. Since both these approaches are mathematically complicated, two alternative simplifications have been developed, namely the mode expansion (ME) approach [4] and the coherent density function (CDF) approach [5]. It has been shown [9, 10] that the characteristic evolution equations within the ME and CDF descriptions are in fact completely equivalent to the CF and Wigner evolution equations. However, the correspondence between the different descriptions and the source conditions, which is a key part of the problem, has not been fully discussed in the literature, although some of the associated problems were mentioned in [9]. Specifically, up to now no regular method has been formulated for how to find the proper ME and CDF for arbitrary source conditions. This means that the applicability of the ME or CDF formalisms has not been completely demonstrated.

The main purpose of the current paper is to present a unified mathematical background for analyzing propagation of partially coherent light excited by an arbitrary source in nonlinear and non-instantaneous media. Emphasis in the analysis is on the formulation of proper evolution equations and source conditions for the propagation of certain functions that characterize the stochastic wave field itself. The presentation relates the existing approaches mentioned above, but also provides the generalizations necessary to justify the ME and CDF approaches. Specifically a regular method is proposed for how to apply the ME and CDF formalisms for arbitrary source conditions and the ambiguity of the choice of ME and CDF formulations for the same source conditions is discussed. The procedure is illustrated by particular examples including an analysis of the interaction between two soliton stripes with different stochastic properties (a soliton stripe is a two-dimensional entity that is uniform in one dimension and soliton shaped in the other). The final result is given in terms of the proper evolution equations whereas the concomitant solutions are not discussed.

2. Mathematical models of partially coherent nonlinear optics and their problems

The present analysis considers propagation of partially coherent light in nonlinear Kerr media, where the basic model is the nonlinear Schrödinger (NLS) equation with an intensity dependent refractive index. It describes the evolution of a slowly varying wave envelope function, $\psi(t, \vec{r}, z)$, as follows,

$$\mathrm{i}\frac{\partial \psi}{\partial z} + \frac{1}{2}\nabla^2\psi + N\psi = 0 \qquad \psi(t, \vec{r}, 0) = \Psi(t, \vec{r}), \tag{1}$$

where z is the normalized distance of propagation, t denotes time measured in a frame moving with the group velocity of the wave, the operator ∇^2 refers to the transverse coordinate $\vec{r}$. $N = N(t, \vec{r}, z)$ accounts for the small nonlinear perturbation of the refractive index and is assumed to depend non-instantaneously on the field intensity $|\psi|^2$. This implies that if the intensity involves fast time variations, the slow medium response will average over these fast temporal fluctuations and N becomes a function of $\langle|\psi|^2\rangle$ only, where $\langle\ \rangle$ denotes time averaging. For a Kerr medium, $N = \langle|\psi|^2\rangle$. Finally, we note that in the form of the NLS equation given by equation (1), dispersive effects (i.e. higher order derivatives in time) have been neglected as compared to the (assumed) dominating diffraction effects.

In general, a theoretical analysis of incoherent light propagation in a nonlinear medium should be based on determining the evolution of the field $\psi(t, \vec{r}, z)$ for a given source function

$\Psi(t, \vec{r})$. However when details of the fast temporal fluctuations of the field are not important, these fluctuations can be considered as a stochastic process, which can be characterized, for instance, by its auto-correlation function (CF)

$$K(t, \vec{r}_1, \vec{r}_2, z) = \langle \psi^*(t, \vec{r}_1, z)\psi(t, \vec{r}_2, z)\rangle.$$

Furthermore, if the source function $\Psi(t, \vec{r})$ represents a stationary stochastic process (with respect to time), the wave field $\psi(t, \vec{r}, z)$ is also a stationary stochastic process at any point in space and its CF as well as the medium refractive index do not depend on time. In this case the use of the CF allows one to present a complete self-consistent description of the propagation of the incoherent light. The evolution of the CF is governed by [2]

$$\begin{aligned} &\mathrm{i}\frac{\partial K}{\partial z} - \frac{1}{2}\nabla^2_{\vec{r}_1} K + \frac{1}{2}\nabla^2_{\vec{r}_2} K - [N(\vec{r}_1, z) - N(\vec{r}_2, z)]K = 0 \\ &N(\vec{r}, z) = K(\vec{r}, \vec{r}, z) \end{aligned} \tag{2}$$

with boundary conditions determined by the source function

$$K(\vec{r}_1, \vec{r}_2, z = 0) = \langle \Psi^*(t, \vec{r}_1)\Psi(t, \vec{r}_2)\rangle. \tag{3}$$

An alternative possibility for giving a self-consistent description of the nonlinear propagation of partially incoherent light is provided by the Wigner function (WF) [8]. The WF and the CF form a Fourier transform pair according to

$$\begin{aligned} &\rho(\vec{r}, \vec{p}, z) = \frac{1}{(2\pi)^l}\int K(\vec{r} + \vec{\xi}/2, \vec{r} - \vec{\xi}/2, z)\exp(\mathrm{i}\vec{p}\cdot\vec{\xi})\,\mathrm{d}^l\vec{\xi} \\ &K(\vec{r}_1, \vec{r}_2, z) = \int \rho\left(\frac{\vec{r}_1 + \vec{r}_2}{2}, \vec{p}, z\right)\exp[\mathrm{i}(\vec{p}\cdot(\vec{r}_2 - \vec{r}_1)]\,\mathrm{d}^l\vec{p}, \end{aligned} \tag{4}$$

where l denotes the dimension of the transverse coordinate space. The boundary condition for the WF is determined by the source function whereas its evolution equation is given by [8]

$$\begin{aligned} &\frac{\partial\rho}{\partial z} + \vec{p}\cdot\nabla_{\vec{r}}\rho + 2N(\vec{r}, z)\sin\left(\frac{1}{2}\overleftarrow{\nabla}_{\vec{r}}\overrightarrow{\nabla}_{\vec{p}}\right)\rho = 0 \\ &N(\vec{r}, z) = \int \rho(\vec{r}, \vec{p}, z)\,\mathrm{d}^l\vec{p}. \end{aligned} \tag{5}$$

Here the sine operator should be interpreted in terms of its expansion and the arrows indicate the direction of operation of the derivatives in its argument.

The evolution equations (2) and (5) are too complicated even for the numerical solution. Therefore, in the literature great attention has been paid to a search for some particular simplifications of the problem using different representations of the correlation function. Specifically, within the mode expansion (ME) approach, the CF is represented as a discrete sum

$$K(\vec{r}_1, \vec{r}_2, z) = \sum_n F_n^*(\vec{r}_1, z)F_n(\vec{r}_2, z), \tag{6}$$

where all mode functions F_n satisfy the same NLS equation (1), but with different boundary conditions. These boundary conditions are the main problem of the ME theory since it is not specified how to determine the boundary conditions for the mode functions F_n in the case of an arbitrary source function $\Psi(t, \vec{r})$. The same problem is associated with the coherent density function (CDF) [5]. Within this approach, the CF is represented by the following integral,

$$K(\vec{r}_1, \vec{r}_2, z) = \int f^*(\vec{r}_1, \vec{\theta}, z)f(\vec{r}_2, \vec{\theta}, z)\exp[\mathrm{i}\vec{\theta}\cdot(\vec{r}_2 - \vec{r}_1)]\,\mathrm{d}^l\vec{\theta}, \tag{7}$$

where the evolution of the CDF, $f(\vec{r}, \vec{\theta}, z)$, is governed by the equation

$$\mathrm{i}\frac{\partial f}{\partial z} + \mathrm{i}\vec{\theta}\cdot\nabla f + \frac{1}{2}\nabla^2 f + Nf = 0, \qquad N(\vec{r}, z) = \int |f(\vec{r}, \vec{\theta}, z)|^2\, \mathrm{d}^l\vec{\theta}. \tag{8}$$

Typically, the CDF is treated as a pure auxiliary function and it has never been shown how to find the CDF corresponding to an arbitrary source function.

Thus the current situation with the ME and CDF approaches is quite unsatisfactory. The evolution equations (1) and (8) are simpler and more useful for numerical simulations than equations (2) or (5). However the necessary boundary conditions are not formulated for general sources and there is no rigorous justification that the ME or CDF are generally applicable to describe the CF of partially coherent light.

3. Justification of the modal expansion and coherent density function approaches

The purpose of the present section is to introduce the fundamental mathematical aspects and concepts involved in an analysis of the NLS equation (1) and to discuss their relation to the different approaches used to simplify the analysis of nonlinear propagation of partially coherent light.

As mentioned above, when the source function $\Psi(t, \vec{r})$ is a stationary stochastic process (with respect to time), the refractive index of the medium does not depend on time. This makes it possible to separate the time dependence of the wave field by expanding the condition at $z = 0$ in a suitable set of basis (or modal) functions, $\{\Phi_n(\vec{r})\}$, according to, cf [11],

$$\psi(t, \vec{r}, 0) = \Psi(t, \vec{r}) = \sum_n C_n(t)\Phi_n(\vec{r}) \tag{9}$$

and then write

$$\psi(t, \vec{r}, z) = \sum_n C_n(t)\varphi_n(\vec{r}, z), \tag{10}$$

where each function, φ_n satisfies a NLS equation of the same form as equation (1) i.e.

$$\begin{aligned} &\mathrm{i}\frac{\partial \varphi_n}{\partial z} + \frac{1}{2}\nabla^2\varphi_n + N\varphi_n = 0 \\ &\varphi_n(\vec{r}, 0) = \Phi_n(\vec{r}). \end{aligned} \tag{11}$$

The nonlinear change in the refractive index, which is determined by the averaged field intensity, can be expressed as

$$N = \langle|\psi|^2\rangle = K(\vec{r}, \vec{r}, z), \tag{12}$$

where K denotes the CF, which, using the expansion of $\psi(t, \vec{r}, z)$, can be expressed in terms of the functions φ_n as

$$K(\vec{r}_1, \vec{r}_2, z) = \langle\psi^*(t, \vec{r}_1, z)\psi(t, \vec{r}_2, z)\rangle = \sum_{mn} A_{nm}\varphi_n^*(\vec{r}_1, z)\varphi_m(\vec{r}_2, z). \tag{13}$$

We emphasize that the correlation matrix A_{nm} does not depend on time and is determined by the coefficients of the expansion (9):

$$A_{nm} = \langle C_n^*(t)C_m(t)\rangle. \tag{14}$$

Thus, equation (11) together with equations (12) and (13) constitute the most general approach to a self-consistent description of the nonlinear propagation of partially coherent light using the representation of the CF in terms of a discrete sum. Within this approach the boundary conditions are completely determined in contrast to the ME theory which considers only the very particular case corresponding to the unit correlation matrix, i.e. when $A_{nm} = \delta_{nm}$.

We emphasize that the same wave field $\Psi(t, \vec{r})$ can be expressed in terms of different basis functions $\{\Phi_n(\vec{r})\}$. Certainly the functions $\varphi_n(\vec{r}, z)$, as well as the expansion coefficients $C_n(t)$ and thereby the correlation matrix A_{nm}, depend on the choice of the set of basis functions. Therefore it is convenient to look for a basis which makes the correlation matrix equal to the unit matrix. A solution of this problem for arbitrary source function $\Psi(t, \vec{r})$ would provide a justification of the ME approach and give a possibility for determining the necessary boundary conditions. As shown below a solution of this problem always exists and is not even unique, i.e. the same wave field can be described within the ME approach using quite different mode representations.

Actually, since the correlation matrix, A_{nm}, is Hermitian, there always exists a unitary transformation, $\tilde{A} = U \cdot A \cdot U^{-1}$ or $\tilde{A}_{nm} = \sum_{kr} U_{nk} A_{kr} U^*_{mr}$ (where U is a unitary matrix for which the components satisfy the equalities $\sum_k U_{nk} U^*_{mk} = \delta_{nm}$) that transforms this matrix to a diagonal form with $\tilde{A}_{nm} = \alpha_n \delta_{nm}$, where α_n denote the eigenvalues of the original correlation matrix A_{nm}. All these eigenvalues are real and non-negative since the CF is positive definite. Correspondingly, a transformation to the new basis functions $\{\tilde{\varphi}_n\}$

$$\tilde{\varphi}_n = \sum_m U_{nm} \varphi_m \tag{15}$$

reduces the representation, equation (13), to its normal form:

$$K(\vec{r}_1, \vec{r}_2, z) = \sum_n \alpha_n \tilde{\varphi}^*_n(\vec{r}_1, z) \tilde{\varphi}_n(\vec{r}_2, z). \tag{16}$$

The transition from the normal to the canonical or ME representation of the CF, equation (6), corresponds to basis functions given by

$$F_n = \sqrt{\alpha_n} \tilde{\varphi}_n. \tag{17}$$

Evidently, the functions $\{\tilde{\varphi}_n\}$, as well as the functions $\{F_n\}$, satisfy the same evolution equations as $\{\varphi_n\}$, but with different boundary conditions determined by (9) and transformations (15) and (17). Thus the above results justify the applicability of the ME approach in general cases and present a regular way for finding the necessary boundary conditions for the modes.

It should be emphasized that the number of nonzero terms in the modal expansion can be less than the number of the original basis functions $\{\Phi_n\}$ if some of the eigenvalues α_n are equal to zero. If so, this means that there are correlations in the excitation of some of the original basis functions. Furthermore, it is evident that the modal expansion corresponding to a given source function $\Psi(t, \vec{r})$ is not unique since any transformation

$$g_n(\vec{r}, z) = \sum_m \tilde{U}_{nm} F_m(\vec{r}, z) \tag{18}$$

based on an arbitrary unitary matrix $\tilde{U}_{nm}$ will result in the same canonical or ME representation of the CF, but in terms of the new generating functions $\{g_n\}$ which satisfy the same NLS equations as the ME functions $\{F_n\}$.

However, among different sets of generating functions there is only one set for which the functions are mutually orthogonal. To find this unique set of generating functions one should start with an original basis $\{\Phi_n\}$ that is mutually orthogonal and normalized to unity. In this case the transformation (15) results in functions $\{\tilde{\varphi}_n\}$ which also are mutually orthogonal and normalized to unity. In fact these functions are the eigenfunctions of the following Fredholm integral equation of the second kind, cf [12, 13]:

$$\alpha_n \tilde{\varphi}_n(\vec{r}_2, z) = \int K(\vec{r}_1, \vec{r}_2, z) \tilde{\varphi}_n(\vec{r}_1, z) \, \mathrm{d}^l \vec{r}_1. \tag{19}$$

The corresponding functions, $\{\tilde{g}_n = \sqrt{\alpha_n}\tilde{\varphi}_n\}$, will be mutually orthogonal (but not normalized to unity) and can be taken as the unique set of generating functions mentioned above.

Thus, an analysis of the propagation of partially coherent light can be based on an expansion of the field $\psi(t, \vec{r}, z) = \sum_n G_n(t) g(\vec{r}, z)$ in terms of generating functions $\{g_n\}$ which can be taken as mutually orthogonal. This makes it possible to use the multi-mode approach, [4], even in cases when the excitation of the source modes, $\Phi_n(\vec{r})$ is not completely uncorrelated. Furthermore, in the latter case, the above considerations provide a regular way of finding the generating functions by diagonalization of the matrix A_{mn} or by solving the eigenvalue problem given by equation (19). However, the problem of finding a suitable set of generating functions may prove difficult for general initial conditions, $\Psi(t, \vec{r})$. Again we emphasize that the choice of orthogonal modes for the expansion given in equation (9) is not unique and neither is the choice of uncorrelated modes. Only the choice of modes which are at the same time orthogonal and uncorrelated is unique.

Instead of the discrete expansion procedure presented above, it may be useful (or even necessary) to use a continuous Fourier expansion, i.e. to write

$$\Psi(t, \vec{r}) = \int C(\vec{\theta}, t)\Phi(\vec{\theta}, \vec{r})\, \mathrm{d}^l\vec{\theta}, \tag{20}$$

where the Fourier harmonics, $\Phi(\vec{\theta}, \vec{r}) = (1/\sqrt{2\pi})^l \exp(\mathrm{i}\vec{\theta}\cdot\vec{r})$, represent a closed basis which is orthogonal and normalized, i.e. $\int \Phi^*(\vec{\theta}_1, \vec{r})\Phi(\vec{\theta}_2, \vec{r})\mathrm{d}^l\vec{r} = \delta(\vec{\theta}_1 - \vec{\theta}_2)$.

The different steps in the subsequent continuous analysis are analogous to those of the discrete analysis presented above. The wave field is now expressed as, cf equation (10)

$$\psi(t, \vec{r}, z) = \int C(\vec{\theta}, t)\varphi(\vec{\theta}, \vec{r}, z)\, \mathrm{d}^l\vec{\theta} \tag{21}$$

where $\varphi(\vec{\theta}, \vec{r}, 0) = \Phi(\vec{\theta}, \vec{r})$ and the function, $\varphi(\vec{\theta}, \vec{r}, z)$, satisfies the same NLS equation as $\varphi_n(\vec{r}, z)$, with the averaged field intensity expressed in terms of the mutual correlation function, which is given by

$$K(\vec{r}_1, \vec{r}_2, z) = \iint A(\vec{\theta}_1, \vec{\theta}_2)\varphi^*(\vec{\theta}_1, \vec{r}_1, z)\varphi(\vec{\theta}_2, \vec{r}_2, z)\, \mathrm{d}^l\vec{\theta}_1\, \mathrm{d}^l\vec{\theta}_2. \tag{22}$$

The correlation properties are defined by

$$A(\vec{\theta}_1, \vec{\theta}_2) = \langle C^*(\vec{\theta}_1, t)C(\vec{\theta}_2, t)\rangle, \tag{23}$$

and similarly to the discrete case, $A(\vec{\theta}_1, \vec{\theta}_2)$ can be made diagonal using a unitary transformation $\varphi(\vec{\theta}, \vec{r}, z) \to \tilde{\varphi}(\vec{\theta}, \vec{r}, z)$, which implies that the correlation function can be written in the following quadratic form:

$$K(\vec{r}_1, \vec{r}_2, z) = \int \alpha(\vec{\theta})\tilde{\varphi}^*(\vec{\theta}, \vec{r}_1, z)\tilde{\varphi}(\vec{\theta}, \vec{r}_2, z)\, \mathrm{d}^l\vec{\theta} \tag{24}$$

or in terms of the generating functions $(\tilde{g}(\vec{\theta}, \vec{r}, z) = \sqrt{\alpha(\vec{\theta})}\tilde{\varphi}(\vec{\theta}, \vec{r}, z))$:

$$K(\vec{r}_1, \vec{r}_2, z) = \int \tilde{g}^*(\vec{\theta}, \vec{r}_1, z)\tilde{g}(\vec{\theta}, \vec{r}_2, z)\, \mathrm{d}^l\vec{\theta}. \tag{25}$$

It should be noted that starting from the orthogonal and normalized Fourier basis $\Phi(\vec{\theta}, \vec{r})$, a unitary transformation will lead to a normalized and orthogonal basis $\tilde{\varphi}(\vec{\theta}, \vec{r}, z)$ and subsequently to the concomitant orthogonal generating function $\tilde{g}(\vec{\theta}, \vec{r}, z)$. However the generating functions are not normalized and therefore an application of one more unitary transformation $(\tilde{g}(\vec{\theta}, \vec{r}, z) \to g(\vec{\theta}, \vec{r}, z))$ results in the same canonical form of the CF, equation (25), although with $g(\vec{\theta}, \vec{r}, z)$ being nonorthogonal. This means that similarly to the discrete case, the generating functions and the representation given by equation (25) are not unique.

The evolution of any generating function is governed by the NLS equation with the intensity expressed in terms of the correlation function.

This continuous description of partially coherent wave propagation in nonlinear media can be used to justify the introduction of the CDF [5] and to determine its relation to the source function. Actually, it is clear from the above analysis that the function defined by

$$f(\vec{r},\vec{\theta},z) = \tilde{g}(\vec{r},\vec{\theta},z)\exp\left(-\mathrm{i}\vec{\theta}\cdot\vec{r} + \mathrm{i}\tfrac{1}{2}\vec{\theta}^2 z\right) \tag{26}$$

possesses all the properties of the CDF. Thus, the regular procedure for determining generating functions consequently also gives a regular procedure for determining the CDF corresponding to any given source field. On the other hand, the non-uniqueness of the generating function implies the possibility of using different CDFs for describing the same wave field.

4. Examples

In this section we present some examples that illustrate the procedure and concepts discussed above.

Example 1. Consider first a wave field which at $z = 0$ is given by $\Psi(t,\vec{r}) = C_1(t)\Phi_1(\vec{r}) + C_2(t)\Phi_2(\vec{r})$ where however Φ_1 and Φ_2 are neither normalized nor mutually orthogonal. The corresponding correlation matrix is given by

$$A_{nm} = \langle C_n^*(t)C_m(t)\rangle = \begin{pmatrix} 1 & R \\ R^* & 1 \end{pmatrix} \tag{27}$$

where R is the correlation coefficient $(0 \leqslant |R| \leqslant 1)$ between $C_1(t)$ and $C_2(t)$. When $R = 0$, the procedure for solving the propagation problem using the ME approach is evident—it is sufficient to take two modal functions $F_1(\vec{r},z)$, $F_2(\vec{r},z)$ which satisfy the boundary conditions $F_n(\vec{r},0) = \Phi_n(\vec{r})$. However, in the case when $R \neq 0$, the original formulation of the ME theory does not give a direct prescription for how to solve the problem. On the other hand, the analysis presented here directly prescribes how to find the boundary conditions for the proper generating functions. Actually, if $R \neq 0$, the unitary transformation

$$\begin{aligned} \tilde{\Phi}_1(\vec{r}) &= a^*\Phi_1(\vec{r}) + a\Phi_2(\vec{r}) \\ \tilde{\Phi}_2(\vec{r}) &= a^*\Phi_1(\vec{r}) - a\Phi_2(\vec{r}) \end{aligned} \tag{28}$$

with $a = \sqrt{R/(2|R|)}$ makes it possible to represent the condition at $z = 0$ in terms of the new functions

$$\Psi(t,\vec{r}) = \tilde{C}_1(t)\tilde{\Phi}_1(\vec{r}) + \tilde{C}_2(t)\tilde{\Phi}_2(\vec{r}), \tag{29}$$

where the coupling matrix now is diagonal and given by

$$\tilde{A}_{nm} = \langle \tilde{C}_n^*(t)\tilde{C}_m(t)\rangle = \begin{pmatrix} 1+|R| & 0 \\ 0 & 1-|R| \end{pmatrix}. \tag{30}$$

Therefore the new functions $\tilde{\Phi}_{1,2}$ can be taken as boundary conditions for the two generating functions $g_1(\vec{r},0) = \sqrt{1+|R|}\tilde{\Phi}_n(\vec{r})$ and $g_2(\vec{r},0) = \sqrt{1-|R|}\tilde{\Phi}_n(\vec{r})$. Clearly, in the case of strong correlation $(|R| = 1)$, there is only one nontrivial generating function. It is also possible to state that if $|R| \neq 1$, then any unitary transformation applied to the generating functions $g_{1,2}$ will result in a pair of new generating functions and any set of such generating functions can be taken as the modes, $F_{1,2}$, within the ME approach.

Example 2. The next example can be considered as a generalization of the previous one to the case of a continuous expansion of the source field. Although, in principle, the continuous

Fourier expansion (20) can be used for any initial condition, $\Psi(t, \vec{r})$, the CDF approach usually deals with only a very simple particular case which can be expressed as follows [1],

$$\Psi(\vec{r}, t) = M(\vec{r}) \int C(\vec{\theta}, t) \exp(\mathrm{i}\vec{\theta} \cdot \vec{r})\, \mathrm{d}^l\vec{\theta}, \tag{31}$$

where the expansion coefficients $C(\vec{\theta}, t)$ are delta-correlated, i.e. $A(\vec{\theta}_1, \vec{\theta}_2) = \langle C^*(\vec{\theta}_1, t) C(\vec{\theta}_2, t)\rangle = J(\vec{\theta}_1)\delta(\vec{\theta}_1 - \vec{\theta}_2)$. The source function (31) represents a deterministic modulation (a modulation described by the function $M(\vec{r})$) of a spatially uniform stochastic field with a spatial spectral intensity given by $J(\vec{\theta})$. According to (22)–(25) the unmodulated field can be described by a generating function which at $z = 0$ satisfies the condition $g(\vec{\theta}, \vec{r}, 0) = \sqrt{J(\vec{\theta})} \exp(\mathrm{i}\vec{\theta} \cdot \vec{r})$. The corresponding boundary condition for the CDF is given by $f(\vec{\theta}, \vec{r}, 0) = \sqrt{J(\vec{\theta})}$ (cf (26)). When $M(\vec{r})$ is not a constant, the source function (31) does not represent a complete Fourier expansion. Nevertheless the CF of the modulated field can easily be expressed in a canonical form by using a generating function satisfying the boundary condition $\tilde{g}(\vec{\theta}, \vec{r}, 0) = \sqrt{J(\vec{\theta})} M(\vec{r}) \exp(\mathrm{i}\vec{\theta} \cdot \vec{r})$ which corresponds to the CDF satisfying the condition $f(\vec{\theta}, \vec{r}, 0) = \sqrt{J(\vec{\theta})} M(\vec{r})$, cf [1].

The situation concerning the boundary conditions is similarly unclear in the case of the CDF approach if at $z = 0$ the field is generated by several partially correlated sources. Let us consider, for example, a case when there are two such sources and the light from each source is modulated independently, i.e. the boundary condition can be written as, cf [14]:

$$\begin{aligned} \Psi(\vec{r}, t) &= \Psi_1(\vec{r}, t) + \Psi_2(\vec{r}, t) \\ \Psi_k(\vec{r}, t) &= M_k(\vec{r}) \int C_k(\vec{\theta}, t) \exp(\mathrm{i}\vec{\theta} \cdot \vec{r})\, \mathrm{d}^l\vec{\theta}, \end{aligned} \tag{32}$$

where the statistical properties of the expansion coefficients are given by

$$\begin{aligned} A_{11}(\vec{\theta}_1, \vec{\theta}_2) &= \langle C_1^*(\vec{\theta}_1, t) C_1(\vec{\theta}_2, t)\rangle = J(\vec{\theta}_1)\delta(\vec{\theta}_1 - \vec{\theta}_2) \\ A_{22}(\vec{\theta}_1, \vec{\theta}_2) &= \langle C_2^*(\vec{\theta}_1, t) C_2(\vec{\theta}_2, t)\rangle = J(\vec{\theta}_1)\delta(\vec{\theta}_1 - \vec{\theta}_2) \\ A_{12}(\vec{\theta}_1, \vec{\theta}_2) &= \langle C_1^*(\vec{\theta}_1, t) C_2(\vec{\theta}_2, t)\rangle = J(\vec{\theta}_1) R(\vec{\theta}_1)\delta(\vec{\theta}_1 - \vec{\theta}_2) \\ A_{21}(\vec{\theta}_1, \vec{\theta}_2) &= A_{12}^*(\vec{\theta}_1, \vec{\theta}_2). \end{aligned} \tag{33}$$

When the correlation factor, $R(\vec{\theta}) = 0$, the correlation function can be expressed in a canonical form in terms of two generating functions, $\tilde{g}_1(\vec{\theta}, \vec{r}, z)$ and $\tilde{g}_2(\vec{\theta}, \vec{r}, z)$, which satisfy the conditions

$$\tilde{g}_k(\vec{\theta}, \vec{r}, 0) = \sqrt{J(\vec{\theta})} M_k(\vec{r}) \exp(\mathrm{i}\vec{\theta} \cdot \vec{r}). \tag{34}$$

The corresponding coherent density functions $f_k(\vec{\theta}, \vec{r}, z)$ are given by

$$f_k(\vec{\theta}, \vec{r}, 0) = \sqrt{J(\vec{\theta})} M_k(\vec{r}). \tag{35}$$

However, in the case when the correlation factor $R(\vec{\theta})$ does not vanish, the matrix $A_{kn}(\vec{\theta}_1, \vec{\theta}_2)$ becomes non-diagonal, and to obtain the canonical form of the CF one needs a unitary transformation similar to that given by equation (28), namely

$$\begin{aligned} \tilde{M}_1(\vec{r}, \vec{\theta}) &= u(\vec{\theta})^* M_1(\vec{r}) + u(\vec{\theta}) M_2(\vec{r}) \\ \tilde{M}_2(\vec{r}, \vec{\theta}) &= u(\vec{\theta})^* M_1(\vec{r}) - u(\vec{\theta}) M_2(\vec{r}) \end{aligned} \tag{36}$$

where $u(\vec{\theta}) = \sqrt{R(\vec{\theta})/|R(\vec{\theta})|}$. This transformation implies that the conditions on the generating and coherent density functions become

$$\tilde{g}_{1,2}(\vec{\theta},\vec{r},0) = \sqrt{1 \pm |R(\vec{\theta})|}\sqrt{J(\vec{\theta})}\tilde{M}_{1,2}(\vec{\theta},\vec{r})\exp(\mathrm{i}\vec{\theta}\cdot\vec{r})$$
$$\tilde{f}_{1,2}(\vec{\theta},\vec{r},0) = \sqrt{1 \pm |R(\vec{\theta})|}\sqrt{J(\vec{\theta})}\tilde{M}_{1,2}(\vec{\theta},\vec{r}), \tag{37}$$

where the plus and minus signs correspond to indices 1 and 2 respectively.

Example 3. Considerable interest recently has been devoted to the dynamics of partially coherent 'soliton stripes', which are structures that at $z = 0$ have a soliton modulation in the x-direction and contain stochastic Fourier components in the y-direction, i.e. they are of the form $\Psi(t,x,y,0) = M(x)\int C(\theta,t)\exp(\mathrm{i}\theta y)\mathrm{d}\theta$ (cf equation (31)). In this example, we will consider the mathematical formulation of the interaction between two co-propagating soliton stripes [14] and show that the problem can be reduced to a study of the interaction between completely coherent vector waves (for a recent analysis of such a system, see, e.g., [15]). Consider the total field at $z = 0$ as given by

$$\Psi(t,x,y) = \sum_{n=1}^{2} M_n(x)\int C_n(\theta,t)\exp(\mathrm{i}\theta y)\,\mathrm{d}\theta, \tag{38}$$

where the coefficients, $C_n(\theta,t)$, satisfy equation (33). In this case the average wave intensity and consequently also the refractive index of the medium are independent of the coordinate y, which makes it possible to separate the dependence of y in the wave field, by writing

$$\psi(t,x,y,z) = \sum_{n=1}^{2}\phi_n(x,z)\int C_n(\theta,t)\exp\left(\mathrm{i}\theta y - \frac{\mathrm{i}\theta^2}{2}z\right)\mathrm{d}\theta. \tag{39}$$

Each function, $\phi_n(x,z)$, satisfies a one-dimensional NLS equation of the form

$$\mathrm{i}\frac{\partial\phi_n}{\partial z} + \frac{1}{2}\frac{\partial^2\phi_n}{\partial x^2} + N\phi_n = 0, \tag{40}$$

where

$$N = \langle|\psi|^2\rangle = \sum_{n,k=1}^{2} B_{kn}\phi_k^*(x,z)\phi_n(x,z)$$
$$\phi_n(x,0) = M_n(x). \tag{41}$$

The matrix B_{kn} is determined by the spectral intensity, $J(\theta)$, and the correlation coefficient, $R(\theta)$, according to

$$B_{11} = B_{22} = \int J(\theta)\mathrm{d}\theta$$
$$B_{12} = \int J(\theta)R(\theta)\,\mathrm{d}\theta = B_{21}^*. \tag{42}$$

This reduces the original problem to that considered in example 2. Using a transformation similar to that given in equation (28), it is possible to present the average intensity, i.e. the correlation function, $K(x,x,y,y,z) = N(x,z)$, in its canonical form:

$$N(x,z) = |\tilde{g}_1(x,z)|^2 + |\tilde{g}_2(x,z)|^2, \tag{43}$$

where the generating functions are given by

$$\tilde{g}_1 = \sqrt{B_{11} + |B_{12}|}\tilde{\phi}_1$$
$$\tilde{g}_2 = \sqrt{B_{11} - |B_{12}|}\tilde{\phi}_2, \tag{44}$$

and

$$\tilde{\phi}_1(x,z) = b^*\phi_1(x,z) + b\phi_2(x,z)$$
$$\tilde{\phi}_2(x,z) = b^*\phi_1(x,z) - b\phi_2(x,z)], \tag{45}$$

with $b = \sqrt{B_{12}/2|B|_{12}|}$. It should be noted that both generating functions can be different from zero even if the original Fourier harmonics are completely correlated (i.e. when $|R(\theta)| = 1$). In such a case the correlation factor will oscillate while fulfilling the inequality $|B_{12}| < B_{11}$.

Obviously, the generating functions $\tilde{g}_n$ are governed by the same NLS equation (40) as the original functions $\phi_n(x,z)$ but they satisfy different boundary conditions which are determined by the transformations (44) and (45). In a general case, the resulting equations are equivalent to those describing coherent propagation of a two-component vector field in a nonlinear Kerr medium with the generating functions, $\tilde{g}_{1,2}$, playing the role of the vector components.

5. Conclusions

Most efforts to describe nonlinear propagation of partially coherent light rely, directly or indirectly, on the basic description of a stochastic process in terms of its correlation function. The present analysis has considered in detail the relation between the correlation function and the different characteristic functions, which have been used for describing the propagation of partially coherent light in nonlinear and non-instantaneous media, i.e. the modal expansion, the coherent density function, the correlation function and the Wigner function approaches. Special emphasis has been given to the problem of relating the source conditions of the stochastic field to the corresponding source conditions for the characteristic functions used in the different approaches and to the problem of the uniqueness of these functions. Specifically, we demonstrate that the coherent density function is not uniquely determined by the stochastic properties of the light field, but rather plays the role of a convenient auxiliary or generating function. It is also shown how to generalize the coherent density function approach to situations where the light field emanates from several sources, which do not necessarily have the same stochastic properties. Actually, for *n* partially correlated sources of coherent light, the coherent density approach must be generalized to include one coherent density function for each of the *n* different sources.

The generality of the approach is illustrated by several examples and e.g. basic equations, describing the interaction of two soliton stripes emanating from two independent, but partially correlated, sources are derived in terms of two nonlinearly coupled vector equations for the corresponding generating functions.

References

[1] Kivshar Yu S and Agrawal G P 2003 *Optical Solitons: From Fibers to Photonic Crystals* (San Diego, CA: Academic)
[2] Pasmanik G A 1974 Self-action of incoherent light beams *Sov. Phys.—JETP* **39** 234–8
[3] Mitchell M, Chen Z, Shih M and Segev M 1996 Self-trapping of partially spatially incoherent light *Phys. Rev. Lett.* **77** 490–3
[4] Mitchell M, Segev M, Coskun T H and Christodoulides D N 1997 Theory of self-trapped spatially incoherent light beams *Phys. Rev. Lett.* **79** 4990–3
[5] Christodoulides D N and Coskun T H 1997 Theory of incoherent self-focusing in biased photorefractive media *Phys. Rev. Lett.* **78** 646–9
[6] Snyder A W and Mitchell D J 1998 Big incoherent solitons *Phys. Rev. Lett.* **80** 1422–4
[7] Shkunov V V and Anderson D Z 1998 Radiation transfer model of self-trapping spatially incoherent radiation by nonlinear media *Phys. Rev. Lett.* **81** 2683–6

[8] Hall B, Lisak M, Anderson D, Fedele R and Semenov V 2002 Statistical theory for incoherent light propagation in nonlinear media *Phys. Rev.* E **65** 035602

[9] Christodoulides D N, Eugenieva E D, Coskun T H, Segev M and Mitchell M 2001 *Phys. Rev.* E **63** 35601

[10] Lisak M, Helczynski L and Anderson D 2003 Relation between the Wigner formalism and the three approaches describing partially incoherent wave propagation in nonlinear optical media *Opt. Commun.* **220** 321–3

[11] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)

[12] Pogarzelski W 1966 *Integral Equations and their Applications* vol 1 (Oxford: Pergamon)

[13] Bastiaans M J 1986 Application of the Wigner distribution function to partially coherent light *J. Opt. Soc. Am.* A **3** 1227–37

[14] Ku T-S, Shih M-F, Sukhorukov A A and Kivshar Yu 2005 Coherence controlled soliton interactions *Phys. Rev. Lett.* **94** 063904

[15] Yang J 2002 *Phys. Rev.* E **65** 036606